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# NEW MINIMAL BOUNDS FOR THE DERIVATIVES OF RATIONAL BÉZIER PATHS AND RATIONAL RECTANGULAR BÉZIER SURFACES

H. E. BEZ AND N. BEZ

ABSTRACT. New minimal bounds are derived for the magnitudes of the derivatives of the rational Bézier paths and the rational rectangular Bézier surface patches of arbitrary degree, which improve previous work of this type in many cases. Moreover, our new bounds are explicitly given by simple and closed-form expressions. An important advantage of the closed-form expressions is that they allow us to prove that our bounds are sharp under certain well-defined conditions. Some numerical examples, highlighting the potential of the new bounds in providing improved estimates, are given in an appendix.

## 1. INTRODUCTION

For vertices  $v = (v_0, \dots, v_n)$  and weights  $w = (w_0, \dots, w_n)$ , for  $w_i > 0$ , let  $\sigma[v, w]$  be the associated rational Bézier path of degree  $n$ . Some classical derivative bounds of Floater [5] are

$$(1) \quad \|\sigma'[v, w]\|_\infty \leq n\Delta_1(v) \left( \frac{\max_i w_i}{\min_j w_j} \right)^2$$

and

$$(2) \quad \|\sigma'[v, w]\|_\infty \leq n\Delta_2(v) \left( \frac{\max_i w_i}{\min_j w_j} \right),$$

where  $\Delta_1(v) = \max_i \|v_{i+1} - v_i\|$  and  $\Delta_2(v) = \max_{i,j} \|v_i - v_j\|$ . Zheng [14] considered the problem of identifying the value of  $\lambda > 0$  which minimises the quantity

$$\frac{\max_i \lambda^i w_i}{\min_j \lambda^j w_j}.$$

The objective was to improve upon the above derivative bounds of Floater by exploiting the invariance of the curve under Möbius transformations of the form  $t \mapsto \lambda t(1 + (\lambda - 1)t)^{-1}$ , for fixed  $\lambda > 0$ , which transform the weights by  $w_i \mapsto \lambda^i w_i$ . In [14] the optimal Möbius parameter  $\lambda$  is not identified in closed form; it is shown that it is obtainable from

$$\lambda = \exp \left( - \frac{\overline{W}_{j_0} + \underline{W}_{k_0}}{j_0 + k_0} \right),$$

where it is necessary to determine integers  $j_0$  and  $k_0$  for which

$$\frac{j_0 \overline{W}_{j_0} - k_0 \underline{W}_{k_0}}{j_0 + k_0} = \max_{1 \leq j, k \leq n} \left\{ \frac{j \overline{W}_j - k \underline{W}_k}{j + k} \right\}.$$

Here,

$$\overline{W}_k = \max_{k \leq i \leq n} (\log w_i - \log w_{i-k}), \quad \text{and} \quad \underline{W}_k = \min_{k \leq i \leq n} (\log w_i - \log w_{i-k}).$$

In subsequent work, Cai and Wang [2] provided an alternative approach to the above result of Zheng for curves, with the benefit that their approach generalised to rectangular Bézier surfaces. Again, the optimal Möbius parameters were not identified in closed form, and in the surface case, the implicit equations satisfied by the optimal Möbius parameters (see Theorem 3.6 in [2]) are substantially more complex than those in the curve case given above.

Our goal in this paper is to obtain closed form expressions for the optimal Möbius parameters as functions of the weights. We also wish to improve upon the minimal derivative bounds obtained from the work of Zheng and Cai–Wang in terms of the size of the bounds, by obtaining the minimal bounds associated to smaller derivative bounds than those of Floater in (1) and (2). To this end, in this paper we consider the derivative bounds of the form

$$(3) \quad \|\sigma'[v, w]\|_\infty \leq n \Delta_\tau(v) \Lambda_\tau(w),$$

where

$$\Lambda_\tau(w) = \max \left\{ \frac{w_{i+1}}{w_i}, \frac{w_i}{w_{i+1}} : i = 0, \dots, n-1 \right\}^{\gamma_\tau}$$

and  $\gamma_\tau > 0$ . It is known (see [9]) that (3) holds when  $\tau = 2$  and  $\gamma_2 = 1$ , for all  $n \geq 2$ , and these bounds are smaller than those in (2). When  $\tau = 1$  it is conjectured in [8] that (3) also holds with  $\gamma_1 = 1$  for all  $n \geq 2$ ; if true, this means such bounds are also smaller than those in (1). This conjecture has been verified for  $n = 1, 2$ , and is currently open for  $n \geq 3$ ; more precisely, the current best known results on the conjecture (see [12] and [8]) are

$$\gamma_1 = \begin{cases} \frac{n}{3} & n \text{ odd} \\ \frac{n+1}{3} & n \text{ even, } n \neq 6 \\ 2 & n = 6. \end{cases}$$

Our main result is that we compute the minimal bound associated with the estimate in (3) and the analogous minimal bound for rectangular surfaces. Importantly, our approach is elementary, easily applies to the curve case and the surface case, and it yields closed form expressions for the optimal Möbius parameter. Moreover, in many cases (described more precisely below), we obtain smaller bounds than those obtained in [2] and [14].

To state our main results we write  $\tilde{w} \sim w$ , if  $\tilde{w}_i = \lambda^i w_i$  for some  $\lambda > 0$ , and introduce the notation

$$\tilde{\Lambda}_\tau(w) = \min_{\tilde{w} \sim w} \Lambda_\tau(\tilde{w})$$

for the minimal (or invariant) bound associated to a pointwise bound  $\Lambda_\tau(w)$ . For curves, our main result is the following.

**Theorem 1.1.** *If  $\sigma[v, w]$  is a rational Bézier path of degree  $n$  with vertices  $v = (v_0, \dots, v_n)$  and positive weights  $(w_0, \dots, w_n)$ , then the pointwise bound*

$$\|\sigma'[v, w]\|_\infty \leq n\Delta_\tau(v)\Lambda_\tau(w)$$

where

$$\Lambda_\tau(w) = \max_i \left\{ \frac{w_{i+1}}{w_i}, \frac{w_i}{w_{i+1}} \right\}^{\gamma_\tau}$$

for some  $\gamma_\tau > 0$ , has minimal bound  $n\Delta_\tau(v)\tilde{\Lambda}_\tau(w)$ , where

$$\tilde{\Lambda}_\tau(w) = \left( \frac{\max_i \left\{ \frac{w_{i+1}}{w_i} \right\}}{\min_j \left\{ \frac{w_{j+1}}{w_j} \right\}} \right)^{\frac{\gamma_\tau}{2}},$$

which is attained at weights  $(w_0, \lambda w_1, \dots, \lambda^n w_n)$  corresponding to the optimal Möbius parameter

$$\lambda = \left( \min_i \left\{ \frac{w_{i+1}}{w_i} \right\} \max_j \left\{ \frac{w_{j+1}}{w_j} \right\} \right)^{-\frac{1}{2}}.$$

For surfaces, we have the following analogue.

**Theorem 1.2.** *If  $\Sigma[v, w]$  is a rational rectangular Bézier surface of degree  $(m, n)$  with vertices  $v$  and weights  $w = (w_{0,0}, \dots, w_{m,n})$  then a pointwise bound of the form*

$$\|\partial_1 \Sigma[v, w]\|_\infty \leq m\Delta_\tau(v)\Lambda_\tau(w),$$

where

$$\Lambda_\tau(w) = \max_{i,j} \left\{ \frac{w_{i,j}}{w_{i+1,j}}, \frac{w_{i+1,j}}{w_{i,j}} \right\}^{\gamma_\tau}$$

for some  $\gamma_\tau > 0$ , has minimal bound  $m\Delta_\tau(v)\tilde{\Lambda}_\tau(w)$ , where

$$\tilde{\Lambda}_\tau(w) = \left( \frac{\max_{i,j} \left\{ \frac{w_{i+1,j}}{w_{i,j}} \right\}}{\min_{k,\ell} \left\{ \frac{w_{k+1,\ell}}{w_{k,\ell}} \right\}} \right)^{\frac{\gamma_\tau}{2}},$$

which is attained for weights  $\lambda^i w_{i,j}$  corresponding to the optimal Möbius parameter

$$\lambda = \left( \min_{i,j} \left\{ \frac{w_{i+1,j}}{w_{i,j}} \right\} \max_{k,\ell} \left\{ \frac{w_{k+1,\ell}}{w_{k,\ell}} \right\} \right)^{-\frac{1}{2}}.$$

In the next section, we prove Theorems 1.1 and 1.2. In the final section, we prove that our results are sharp in a certain sense; this is made possible precisely because we obtain closed-form expressions for the minimal bounds in our theorems above. Finally, in an Appendix, we present some numerical examples to highlight the gain provided by our results over the previous work in [2] and [14].

## 2. PROOFS OF THEOREMS 1.1 AND 1.2

Our approach is elementary and Theorems 1.1 and 1.2 will follow almost immediately from the following lemma.

**Lemma 2.1.** *If  $x_0, x_1, \dots, x_{n-1} > 0$ , then*

$$\min_{\lambda > 0} \max_i \left\{ \lambda x_i, \frac{1}{\lambda x_i} \right\} = \left( \frac{\max_i \{x_i\}}{\min_j \{x_j\}} \right)^{\frac{1}{2}}$$

*which occurs at*

$$\lambda = (\min_i \{x_i\} \max_j \{x_j\})^{-\frac{1}{2}}.$$

*Proof.* For  $x_i > 0$ , the functions  $f_i^-(\lambda) = \frac{1}{\lambda x_i}$  are monotonically decreasing on  $0 \leq \lambda < \infty$  with  $\lim_{\lambda \rightarrow 0} f_i^-(\lambda) = \infty$  and  $\lim_{\lambda \rightarrow \infty} f_i^-(\lambda) = 0$  and the functions  $f_i^+(\lambda) = \lambda x_i$  are monotonically increasing on  $0 \leq \lambda < \infty$  with  $f_i^+(0) = 0$  and  $\lim_{\lambda \rightarrow \infty} f_i^+(\lambda) = \infty$ .

We write  $x_k = \min_j \{x_j\}$  and  $x_\ell = \max_i \{x_i\}$  (it is not important that the choice of  $k$  and  $\ell$  may not be unique). Then, we have  $\max_i \{f_i^-\} = f_k^-$ ,  $\max_i \{f_i^+\} = f_\ell^+$  and hence

$$\max_i \{f_i^+, f_i^-\} = \max \{f_k^-, f_\ell^+\} = \begin{cases} f_k^- & \text{on } [0, \lambda^*) \\ f_\ell^+ & \text{on } [\lambda^*, \infty). \end{cases}$$

Here,  $\lambda^* = (x_k x_\ell)^{-\frac{1}{2}}$  is the value of  $\lambda$  at which  $f_k^-(\lambda) = f_\ell^+(\lambda)$  and, from the monotonic nature of  $f_k^-$  and  $f_\ell^+$ , also the point at which the minimum of the function  $\max_i \{f_i^+, f_i^-\}$  occurs. It follows that

$$\min_{\lambda > 0} \max_i \left\{ \lambda x_i, \frac{1}{\lambda x_i} \right\} = \left( \frac{x_\ell}{x_k} \right)^{\frac{1}{2}} = \left( \frac{\max_i \{x_i\}}{\min_j \{x_j\}} \right)^{\frac{1}{2}}$$

as claimed.  $\square$

For Theorem 1.1, with  $\tilde{w}_i = \lambda^i w_i$  we have  $\frac{\tilde{w}_{i+1}}{\tilde{w}_i} = \lambda \frac{w_{i+1}}{w_i}$ , and so the result follows by applying Lemma 2.1 with  $x_i = \frac{w_{i+1}}{w_i}$ . For the surface case in Theorem 1.2, weight vectors,  $\tilde{w}$  equivalent to  $w$  are those for which  $\tilde{w}_{i,j} = \lambda^i \mu^j w_{i,j}$ , where  $\lambda, \mu > 0$ . Then

$$\frac{\tilde{w}_{i+1,j}}{\tilde{w}_{i,j}} = \lambda \frac{w_{i+1,j}}{w_{i,j}}$$

and, as with Theorem 1.1, the result follows by applying Lemma 2.1 with  $x_i = \frac{w_{i+1,j}}{w_{i,j}}$ .

*Remarks.* (1). At this stage, we can point out why our approach is significantly simpler than that in [2] and [14], particularly in the surface case. The reason is that in the expression

$$\Lambda_\tau(w) = \max_{i,j} \left\{ \frac{w_{i,j}}{w_{i+1,j}}, \frac{w_{i+1,j}}{w_{i,j}} \right\}^{\gamma_\tau}$$

the second index  $j$  is the same on the numerator and denominator, and as a consequence, the parameter  $\mu$  does not appear in the minimisation problem.

(2). As far as we are aware, the only previous known results on minimal bounds for arbitrary degree are those in [2] and [14]. Comparing our results with those obtained in these papers, we remark that in the case  $\tau = 2$ , our minimal bounds for both curves and surfaces are smaller for all  $n \geq 2$ . When  $\tau = 1$ , since we

have already noted in the Introduction that we can take  $\gamma_1$  to be either 1 or 2 for  $n \leq 6$ , our minimal bounds are smaller for such degrees. We also remark that our minimal bounds will automatically improve subject to any further progress on the conjecture of Li *et al* concerning the minimum value of  $\gamma_1$ . If the conjecture is completely confirmed for all  $n \geq 2$  (or, less strongly, if we can take  $\gamma_1 \leq 2$  for all  $n \geq 2$ ) then our minimal bounds will be better than those obtained in [2] and [14] for the remaining  $n \geq 7$  in the  $\tau = 1$  case.

(3) There are pointwise bounds which improve upon bounds of the form (3); see, for example, some of the derivative bounds obtained in [1], [3]–[7], [9], [11]–[13], and also the useful overview of such bounds in [8]. However, except for the pointwise bounds  $\Lambda_\tau(w)$  under consideration in this paper, the associated *minimal* bounds appear to be difficult to compute for arbitrary degree (especially so in closed-form).

### 3. SHARPNESS OF OUR MINIMAL BOUNDS

In this section we write  $J_i$ , for  $i = 0, \dots, n-2$ , for the independent set of invariants for the rational Bézier paths of degree  $n$  defined by

$$J_i(w) = \frac{w_{i+1}}{(w_i w_{i+2})^{\frac{1}{2}}}.$$

**Theorem 3.1.** *If  $J_i(w) \geq 1$ , for all  $i = 0, \dots, n-2$ , we have*

$$(4) \quad \tilde{\Lambda}_\tau(w) = \left( \prod_{i=0}^{n-2} J_i(w) \right)^{\gamma_\tau}.$$

*Consequently, for such  $w$ , the associated minimal bounds for which  $\gamma_\tau = 1$  are sharp.*

*Proof.* Observe that if  $J_i(w) \geq 1$  then  $\frac{w_{i+1}}{w_i} \geq \frac{w_{i+2}}{w_{i+1}}$  and it follows immediately from Theorem 1.1 that

$$\tilde{\Lambda}_\tau(w) = \left( \frac{w_1 w_{n-1}}{w_0 w_n} \right)^{\frac{\gamma_\tau}{2}}.$$

In addition, it is easy to show that  $(\prod_{i=0}^{n-2} J_i(w))^2 = \frac{w_1 w_{n-1}}{w_0 w_n}$  and therefore (4) holds. For the sharpness claim, we note that

$$|\sigma[v, w]'(0)| = n|v_1 - v_0| \frac{w_1}{w_0} \quad \text{and} \quad |\sigma[v, w]'(1)| = n|v_n - v_{n-1}| \frac{w_{n-1}}{w_n},$$

so by choosing  $v$  such that  $|v_1 - v_0| = |v_n - v_{n-1}| = \Delta_\tau(v)$  we obtain

$$\|\sigma[v, w]'\|_\infty \geq (|\sigma[v, w]'(0)| |\sigma[v, w]'(1)|)^{1/2} = n \Delta_\tau(v) \prod_{i=0}^{n-2} J_i(w).$$

For such  $w$ , it follows that when  $\gamma_\tau = 1$  the associated minimal bounds are sharp.  $\square$

Recall that when  $\tau = 2$  we can take  $\gamma_2 = 1$  for all  $n \geq 2$ , and when  $\tau = 1$  we can take  $\gamma_1 = 1$  for  $n = 2, 3$ .

By modifying the above proof, it can be shown that when  $J_i(w) \leq 1$ , for all  $i = 0, \dots, n-2$  we have

$$\tilde{\Lambda}_\tau(w) = \left( \prod_{i=0}^{n-2} J_i(w) \right)^{-\gamma_\tau},$$

which means such bounds always exhibit singularities near the origin of the invariant space. It would be interesting to find pointwise bounds smaller than  $\Lambda_\tau(w)$ , for which singularities do not occur in this way; the authors [1] have demonstrated in the quadratic path case that this can be done.

## APPENDIX

For comparative purposes, we use the same numerical examples used in [2] to highlight our gain.

For rational Bézier paths of degree 6 with weights  $w = (5, 2, 8, 56, 80, 96, 64)$ :

(i) Using the pointwise bound of Zhang–Ma (corresponding to  $\tau = 1, \gamma_1 = 2$ ) we obtain, from Theorem 1.1, the minimal value  $105\Delta_1(v)$ , whereas [2] (and [14]) yields  $294\Delta_1(v)$ .

(ii) Using the Selimovic bound (corresponding to  $\tau = 2, \gamma_2 = 1$ ) we obtain, from Theorem 1.1, the minimal value  $6\left(\frac{35}{2}\right)^{\frac{1}{2}}\Delta_2(v) < \frac{36}{\sqrt{2}}\Delta_2(v)$ , whereas [2] (and [14]) yields  $42\Delta_2(v)$ .

For the degree (3, 3) Bézier surface with weights:

$$\begin{bmatrix} 5 & 18 & 27 & 27 \\ 2 & 42 & 36 & 270 \\ 8 & 60 & 72 & 216 \\ 32 & 72 & 144 & 216 \end{bmatrix}$$

we obtain  $15\Delta_2(v)$  from Theorem 1.2, whereas [2] yields a bound of  $1029\Delta_2(v)$  for this example.

For the degree (2, 3) Bézier surface with weights:

$$\begin{bmatrix} 1 & 7 & 5 & 3 \\ 12 & 16 & 12 & 4 \\ 8 & 16 & 4 & 8 \end{bmatrix}$$

we obtain  $12\Delta_2(v)$  from Theorem 1.2, whereas [2] yields  $1024\Delta_2(v)$ .

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